Spring 2006

# Dynamic Models in Economics

# Problem Set 1

## Exercise 1 : Inflation and Unemployment

Consider a simple macroeconomic model with three goods : labor, $n$, money, $m$, and a consumption good, $y$. This latter is produced in each period $t = 0, \ldots, \infty$, according to a linear technology : $y_t = \alpha n_t$, $\alpha \in (0,1)$. Let $p_t$ be the price of the consumption good, $y_t^d$ the aggregate demand, $y_t^s$ the aggregate supply, $n^s$ the labor supply, $n_t$ the labor demand, $u_t$ the unemployment rate, and $w_t$ the nominal wage of period $t$. All these variables are linked by the following equations :

$$
\begin{aligned}
y_t^d &= m_t - p_t && (1)\\
n_t &= \frac{1}{1-\alpha}(w_t - p_t) && (2)\\
y_t^s &= \frac{\alpha}{1-\alpha}(p_t - w_t) && (3)\\
u_t &= n^s - n_t && (4)\\
w_{t+1} - w_t &= -\lambda(u_t - \bar{u}) && (5)\\
m_{t+1} - m_t &= \mu, && (6)
\end{aligned}
$$

where $\lambda > 0$, $\mu$, and $\bar{u}$ are exogenous parameters of the model.

1. Comment on equation (5).
2. Determine the macroeconomic equilibrium of period $t$.
3. Derive the laws of motion of production, $y_t$, unemployment rate, $u_t$, and inflation, $\pi_t$.
4. What is the stationary level of unemployment, $u^*$ ? Is $u^*$ a stable stationary equilibrium ?
5. Suppose now that equation (5) is replaced by

$$w_{t+1} - w_t = p_{t+1} - p_t - \lambda(u_t - \bar{u}), \quad (7)$$

Show that, under this new specification of the model, the dynamics of the vector $X \equiv (u, \pi)$ can be represented by a system of linear equations of the form :

$$X_{t+1} = AX_t + F. \tag{8}$$

6. Derive the stationary equilibrium of system (8), and show that it is stable whenever $1 + \alpha > \lambda$. Draw a phasis diagram.

## Exercise 2 : Credit Rationing and Stagflation

Consider a single-good economy. The period-$t$ price of the good is equal to $P_t$. Firms invest in that good in period $t-1$ to produce in period $t$. Let $I_{t-1}$ be the investment in $t-1$, and $Y_t$ the level of production in $t$. We assume that the production function is of the form

$$Y_t = vI_{t-1} \tag{9}$$

where $v > 1$ is a measure of productivity. The demand function is

$$D_t = a + bY_t, \tag{10}$$

where $b \in (0, 1)$. If firms expect to sell $D_{t+1}$ tomorrow, they must invest $D_{t+1}/v$ today. As investment is entirely financed by credit, firms' real demand for credit is then :

$$\frac{B_t^d}{P_t} = \frac{D_{t+1}}{v}. \tag{11}$$

We assume that banks can ration credit by limiting the amount lent at some threshold $B_t^o$. As a consequence, the credit that is actually given to firms is :

$$B_t = \min(B_t^d, B_t^o). \tag{12}$$

The upper-bound $B_t^o$ is given by the following equation :

$$\frac{B_t^o}{P_t} = \frac{L_t}{P_t} + \alpha Y_t, \tag{13}$$

where $L_t$ represents the bank reserves, and is then an index of credit availability. To simplify the analysis, assume that $L_t = L$ for each $t = 0, 1, \ldots$ The level of investment is determined by the credit obtained by firms :

$$I_t = \min\left(\frac{D_{t+1}}{v}, \frac{B_t}{P_t}\right). \tag{14}$$

We finally assume that the price evolves according to the following adjustment process :

$$P_{t+1} - P_t = \lambda(D_t - Y_t),\ \lambda > 0. \tag{15}$$

1. Compute the stationary equilibrium $(I^*, P^*)$ of this model under the assumption that credit is rationed.
2. Assume $\alpha v > 1$. Study the dynamics of the model, and show that there exists $\bar{L} \geq 0$ such that $(I^*, P^*)$ is a saddle-point whenever $L$ exceeds $\bar{L}$.
3. Draw a phasis diagram in the $(I_{t-1}, L/P_t)$-plane, and comment on the trajectories.

## Exercise 3 : A Growth Model with Pollution

A planning agency is interested in the impact of consumption on the level of pollution faced by a society. Let $c_t$ be the society's consumption, and $\pi_t$ the level of pollution measured at date $t$. The planner seeks to determine the pair $(c, \pi)$ that maximizes social welfare under the constraint of pollution accumulation. Formally, the problem $(P)$ to be solved is the following :

$$(P) \left\{ \begin{array}{ll} \max_{\{c_t, \pi_t, t>0\}} & \int_0^{+\infty} e^{-\beta t} \left[ u(c_t) - v(\pi_t) \right] dt \\ \text{s.t.} & \dot{\pi}_t = a c_t - b\pi_t - d \\ & \pi_0 \text{ given} \end{array} \right.$$

where $\beta, a, b, d$ are all positive. $u$ is a function of class $\mathcal{C}^1$, which is strictly increasing and concave in $c_t$, and $v$ is also a function of class $\mathcal{C}^1$, which is strictly increasing and convex in $\pi_t$. We assume that $\lim_{c \to \infty} u'(c) = 0$, and that $v'(0) = 0$.

1. Comment on the equation of problem $(P)$. Defining $\lambda_t$ as the discounted value of the co-state variable, write the Hamiltonian of $(P)$. Is the problem convex ?
2. Show that the following condition must hold at the optimum : $u'(c_t) = -a\lambda_t^* e^{\beta t}$, where $\lambda_t^*$ is measured along every optimal trajectory. Comment on this optimality condition.
3. Using the Hamiltonian system, show that the optimal consumption satisfies the following condition :
$$u''(c_t) \dot{c}_t = (b + \beta) u'(c_t) - a v'(\pi_t) .$$
4. Show that the set of points $(\pi, c)$ such that $\dot{c}_t = 0$ can be represented by a strictly decreasing curve, while the set of points such that $\dot{\pi}_t = 0$ can be represented by a strictly increasing curve. Draw a phasis diagram in the $(\pi, c)$-plane.

5. Show that there exists a unique stationary equilibrium, which is a saddle-point.
6. Write the transversality conditions. What are the characteristics of the optimal trajectories of this problem ?

# Exercise 4 : Capital Taxation

There is an infinitely lived representative household living in a single-good economy. The household likes consumption, leisure streams $\{c_t, l_t\}_{t=0}^{\infty}$ that give higher values of :

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \beta \in (0,1) \tag{16}$$

where $u$ is increasing, strictly concave, and three times continuously differentiable in $c$ and $l$. The household is endowed with one unit of time that can be used for leisure $l_t$ and labor $n_t$ :

$$l_t + n_t = 1.$$

The single good is produced with labor $n_t$ and capital $k_t$ as inputs. The output can be consumed by households, used by the government, or used to augment the capital stock. The technology is described by :

$$c_t + g_t + k_{t+1} = F(k_t, n_t) + (1-\delta)k_t,$$

where $\delta \in (0,1)$ is the rate at which capital depreciates, and $\{g_t\}_{t=0}^{\infty}$ is an exogenous sequence of government purchases. We assume a concave production function $F(k,n)$ that exhibits constant returns to scale.

The government finances its stream of purchases $\{g_t\}_{t=0}^{\infty}$ by levying time-varying taxes on earnings from capital at rate $\tau_t^k$ and from labor at rate $\tau_t^n$. Let $r_t$ and $w_t$ be the market-determined rental rate of capital and the wage rate of labor, respectively, denominated in units of time $t$ goods.

In each period, the representative firm takes $(r_t, w_t)$ as given and rents capital and labor from households to maximize profits :

$$\Pi = F(k_t, n_t) - r_t k_t - w_t n_t.$$

Let $u_c$ be the derivative of $u(c_t, l_t)$ with respect to consumption ; $u_l$ is the derivative with respect to $l$. We use $u_c(t)$ and $F_k(t)$ and so on to denote the time-$t$ values of the indicated objects, evaluated at an allocation to be understood from the context.

**A. Government :** Determine the government's budget constraint in period $t$.

**B. Households :** 1. Determine the household's budget constraint in period $t$.

2. Write the problem of the household. Establish the Bellman equation with the budget constraint as transition law (precise which variable is the control, and which is the state).
3. Deduce from the previous question that the optimal consumption, leisure streams can be written as :

$$c_t = C(k_t) \text{ et } l_t = 1 - N(k_t), \forall t = 0, \ldots, \infty.$$

4. Show that the solutions to the household's problem satisfy :

$$u_l(t) = u_c(t)(1 - \tau_t^n) w_t,$$

$$u_c(t) = \beta u_c(t+1)[(1 - \tau_{t+1}^k) r_{t+1} + 1 - \delta].$$

**C. Firms :** 1. Show that firm's demands of capital and labor satisfy :

$$r_t = F_k(t) \text{ et } w_t = F_n(t).$$

Comment.

2. Using Euler theorem, show that the firm's profit is zero in equilibrium.

**D. The Ramsey problem :** Given the initial capital stock, $k_0$, the Ramsey problem is to choose a taxation stream $\{\tau_t^k, \tau_t^n\}_{t=0}^{\infty}$ that maximizes (16), taking agents' private decisions into account.

1. Write the Ramsey problem in Lagrangian form.
2. Using the first order conditions with respect to $k_{t+1}$ and the answer to question B.4, show that, at the steady state, the capital tax rate is zero.[1]

# Exercise 5 : Optimal Growth and Consumption Habits

Consider the problem of a government that has to choose an investment policy, given households' consumption habits. Its objective function is assu-

[1] We assume here that a steady state exists.

med to be

$$\sum_{t=0}^{+\infty} \beta^t \left(\ln c_t + \gamma \ln c_{t-1}\right), \qquad 0 \;<\; \beta < 1, \quad \gamma > 0$$

$$s.t. \quad c_t + k_{t+1} \;\leq\; Ak_t^{\alpha} \qquad 0 < \alpha < 1, \quad A > 0$$

where $k_0 > 0$ et $c_{-1}$ are given by the current situation.

1. Write the Bellman equation.
2. Show that the solution is of the form : $V(k_t, c_{t-1}) = E + \ln k_t + G \ln c_{t-1}$, and that the optimal policy is of the form $\ln k_{t+1} = I + H \ln k_t$ where $E$, $F$, $G$, $H$, and $I$ are constant parameters. Compute them as functions of $A$, $\alpha$, $\beta$ and $\gamma$. What do you notice about the optimal policy ?
3. Explain why this last result can be extended to the following case :

$$\sum_{t=0}^{+\infty} \beta^t \left(u\left(c_t\right) + \gamma v\left(c_{t-1}\right)\right), \qquad 0 < \beta < 1, \quad \gamma > 0$$

$$s.t. \quad c_t + k_{t+1} \leq f\left(k_t\right) \qquad où\ f'\left(0\right) = +\infty,\ f' > 0,\ et\ f'' < 0.$$

# Exercise 6 : Extraction of a Common Resource

We study here the extraction of a renewable resource by competing players. Let $k^t$ denote the current stock of the common resource. At date $t$, players 1 and 2 simultaneously choose how much to extract ($a_1^t \geq 0$ and $a_2^t \geq 0$). If $k^t \geq a_1^t + a_2^t$, player $i$ gets instantaneous payoff $g_i(a_i^t)$, and the stock at the beginning of date $t+1$ is $k^{t+1} = f(k^t - a_1^t - a_2^t)$, where $f(k) = k^{\alpha}$, $\alpha \in (0,1)$, is the transition or reproduction function. If $k^t < a_1^t + a_2^t$, each player gets $k^t/2$, which yields payoff $g_i(k^t/2)$ and $k^{t+1} = f(0) = 0$.

For a profile $(s_1(\cdot), s_2(\cdot))$ of pure Markov strategies, let $\psi(k) = k - s_1(k) - s_2(k)$ denote the remaining stock at the end of the period. Note that, without loss of generality, one can restrict $s_i$ to belong to $[0, k]$.

1. Show that a differentiable Markov perfect equilibrium must satisfy the Bellman equation[2] : For all $k$,

$$g_i'(s_i(k_t)) = \delta g_i'(s_i(f(\psi(k_t))))f'(\psi(k_t))[1 - s_j'(f(\psi(k_t))]. \qquad (17)$$

2. Assume $g_i(x) = \ln x$, and show that the strategies

$$s_i(k_t) = \frac{1 - \alpha\delta}{2 - \alpha\delta} k_t \ ,\ i = 1, 2,$$

[2] We assume here that a differentiable Markov perfect equilibrium exists.

constitute an MPE. Determine the stationary value of the stock, $\hat{k}$, that results from these strategies.

3. Compute the steady state, $k^*$, of a centrally planned economy, in which a social planner would choose extraction rates so as to maximize a weighted sum of the two players' intertemporal utilities.
4. Compare $\hat{k}$ and $k^*$, and interpret the result.